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Brief paper

Saturated control of an uncertain nonlinear system with input delay☆

N. Fischer [a], A. Dani [a], N. Sharma [b], W.E. Dixon [a,1]

[a] Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL, USA

[b] Department of Mechanical Engineering and Materials Science, University of Pittsburgh, Pittsburgh, PA, USA

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ABSTRACT

This paper examines saturated control of a general class of uncertain nonlinear systems with time-delayed actuation and additive bounded disturbances. The bound on the control is known a priori and can be adjusted by changing the feedback gains. A Lyapunov-based stability analysis utilizing Lyapunov–Krasovskii (LK) functionals is provided to prove uniformly ultimately bounded tracking despite uncertainties in the dynamics. A numerical example is presented to demonstrate the performance of the controller.

1. Introduction

As described in the survey papers (Gu & Niculescu, 2003; Richard, 2003; Sipahi, Niculescu, Abdallah, Michiels, & Gu, 2011; Watanabe, Nobuyama, & Kojima, 1996) (and the hundreds of references therein) and relatively recent monographs such as (Gu, Kharitonov, & Chen, 2003; Krstic, 2009; Loiseau, Michiels, Niculescu, & Sipahi, 2009; Mahmoud, 2000; Niculescu & Gu, 2004), time delays are pervasive in nature and engineered systems. A few well-known and documented engineering applications include digital implementation of a continuous control signal, regenerative chatter in metal cutting (especially prevalent in high-speed manufacturing), delays in torque production due to engine cycle delays in internal combustion engines, chemical process control, rolling mills, teleoperated robotic systems, control over networks, and active queue management (AQM). Delays are also inherent in many biological process such as the delay in a person's response due to drugs and alcohol, delays in force production in muscle, and the cardiovascular control system. Systems that do not compensate for such delays can exhibit reduced performance and potential instability.

Motivated by performance and stability problems with time-delayed systems, and inspired by the classic results of Smith (1959) and Artstein (1982), solutions to the input delay problem typically exploit predictive-based control methods. Several results have used variations of these methods to solve the input delay problem for linear systems with certain and uncertain dynamics (Bresch-Pietri & Krstic, 2009; Gomez, Orlov, & Kolmanovsky, 2007; Gu & Niculescu, 2003; Krstic & Smyshlyaev, 2008; Kwon & Pearson, 1980; Richard, 2003; Yildiz, Annaswamy, Kolmanovsky, & Yanakiev, 2010). However, as stated in the "Beyond this Book" section of the seminal work in Krstic (2009), Krstic indicates that approaches developed for uncertain linear systems do not extend in an obvious way to nonlinear plants since the linear boundedness of the plant model is explicitly used in the stability proof, and that new methods must be developed for select classes of nonlinear systems with input delays. Several results have been developed for input-delayed nonlinear systems with exact model knowledge (Henson & Seborg, 1994; Jankovic, 2006; Krstic, 2008; Mazenc & Bliman, 2006; Teel, 1998), but few results examine the input delay problem for uncertain nonlinear systems. Specifically, recent results in Sharma, Bhasin, Wang, and Dixon (2011) proposed the development of a predictor-based controller for a time-delayed actuation system with parametric uncertainty and/or additive bounded disturbances using Lyapunov–Krasovskii (LK) functionals to achieve a semi-global uniformly ultimately bounded tracking result.

Due to the fact that control input signals are a function of the system states, large initial conditions or unmodeled disturbances

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E-mail addresses: nic.r.fischer@gmail.com (N. Fischer), ashwin31@ufl.edu (A. Dani), nitin2@ualberta.ca (N. Sharma), wdixon@ufl.edu (W.E. Dixon).

[1] Tel.: +1 352 846 1463; fax: +1 352 392 7303.

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may cause the controller to exceed physical limitations. For systems with input delays, errors can build over the delay interval also leading to large actuator demands, exacerbating potential problems with actuator saturation. Because degraded control performance and the potential risk of thermal or mechanical failure can occur when unmodeled actuator constraints are violated, control schemes which can ensure performance while operating within actuator limitations are motivated.

The majority of saturated controllers presently available for systems with input delays are based on linear plant models (Bin & Zongli, 2010; Cao, Wang, & Tang, 2007; Park, Choi, & Choo, 2000; Zhou, Lin, & Duan, 2010), and only a few results are present for nonlinear systems (especially those with uncertainties). In Mazenc et al. (2003), global uniform asymptotic stabilization is obtained with bounded feedback of a strict-feedforward linear system with delay in the control input. The authors were able to extend the result to an uncertain but disturbance-free strict-feedforward nonlinear system with delays in the control input in Mazenc et al. (2004) using a system of nested saturation functions. The controller requires a nonlinear strict-feedforward dynamic system with parametric uncertainty, $h(x)$, which satisfies the following condition: $|h(x_{i+1}, x_{i+2}, \ldots, x_n)| \leq M\left(x_{i+1}^2, x_{i+2}^2, \ldots, x_n^2\right)$, where M denotes a positive real number when $|x_j| \leq 1$, $j = i + 1, \ldots, n$. Unlike compensation-based delay methods, the design in Mazenc et al. (2004) cleverly exploits the inherent robustness to delay in the particular structure of the feedback law and plant. Krstic proposed a saturated compensator-based approach in Krstic (2010), which results in a nonlinear version of the Smith Predictor (Smith, 1959) with nested saturation functions, and is able to achieve quantifiable closed-loop performance by using an infinite-dimensional compensator for strict-feedforward nonlinear systems with no uncertainties.

The work presented in this paper (along with the preliminary work in Fischer, Dani, Sharma, & Dixon, 2011) introduces a new saturated control design that can predict/compensate for input delays in uncertain nonlinear systems. Based on our previous non-saturated feedback work in Sharma et al. (2011), a continuous saturated controller is developed which allows the bound on the control to be known a priori and to be adjusted by changing the feedback gains. The saturated controller is shown to guarantee uniformly ultimately bounded tracking despite a known constant input delay, parametric uncertainties, and sufficiently smooth additive disturbances.[2] Admissible values for the known input delay can be determined based on sufficient gain selection and the initial conditions of the system. The result is based on the idea of developing a delay compensating auxiliary signal to obtain a delay-free open-loop error system and the construction of an LK functional to cancel the time-delayed terms. The result is valid in a domain that is a function of the initial conditions. This domain can be enlarged by selecting larger control gains; however, the size of the domain is ultimately restricted by the saturation limits of the actuator. A numerical simulation is presented to demonstrate the performance of the controller.

2. Dynamic model and properties

Consider a class of nonlinear systems described by

$$\ddot{x} = f(x, \dot{x}, t) + u(t - \tau) + d(t), \tag{1}$$

[2] To implement this controller on an Euler–Lagrange system with an Inertia matrix, the control input will be a function of the Inertia matrix. This fact is discussed in the subsequent simulation section. Thus, to apply the subsequent theory to an Euler–Lagrange system, one would require exact model knowledge of the Inertia matrix. For a control design which features an uncertain Inertia matrix, see Fischer et al. (2011).

where $x(t), \dot{x}(t) \in \mathbb{R}^n$ are the generalized system states, $u(t-\tau) \in \mathbb{R}^n$ represents the generalized delayed control input vector, where $\tau \in \mathbb{R}^+$ is a constant time delay, $f(x, \dot{x}, t) : \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ is an unknown $\mathcal{C}^2$ function that is uniformly bounded in t, and $d(t) \in \mathbb{R}^n$ denotes a sufficiently smooth exogenous disturbance (e.g., unmodeled effects).

The subsequent development is based on the assumption that $x(t)$ and $\dot{x}(t)$ are measurable outputs, the time delay constant, τ, is known, and the control input vector $u(t)$ and its past values (i.e., $u(t-\theta)\ \forall\theta \in [0\ \tau]$) are measurable. Throughout the paper, a time-dependent delayed function is denoted as $\zeta(t-\tau)$ or ζ_τ. Additionally, the following assumptions are used.

Assumption 1. The disturbance term and its first time derivative are bounded by known constants, i.e., $\|d(t)\| \leq c_1$, $\|\dot{d}(t)\| \leq c_2$, where $c_1, c_2 \in \mathbb{R}^+$.

Assumption 2. The desired trajectory $x_d(t) \in \mathbb{R}^n$ is designed such that $x_d(t), \dot{x}_d(t), \ddot{x}_d(t) \in \mathcal{L}_\infty$.

Assumption 3. The function $f(x, \dot{x}, t)$ satisfies the following inequality: $\|f(x, \dot{x}, t) - f(x_d, \dot{x}_d, t)\| \leq \rho(\|\varphi\|)\|\varphi\|$, where $\varphi(x, \dot{x}, x_d, \dot{x}_d) \in \mathbb{R}^{2n}$ is defined as $\varphi = [x - x_d,\ \dot{x} - \dot{x}_d]^T$ and $\rho(\cdot) : \mathbb{R} \to \mathbb{R}$ is a positive globally invertible function.

Remark 1. Imposing Assumption 3 on $f(x, \dot{x}, t)$ is less restrictive than claiming the function satisfies the global Lipschitz condition (which would yield a linear bound in the states, i.e., $\rho(\|\varphi\|) = \rho$).

To aid the subsequent control design and analysis, the vector $\text{Tanh}(\cdot) \in \mathbb{R}^n$ and the matrix $\text{Cosh}(\cdot) \in \mathbb{R}^{n\times n}$ are defined as follows:

$$\text{Tanh}(\xi) \triangleq [\tanh(\xi_1), \ldots, \tanh(\xi_n)]^T, \tag{2}$$

$$\text{Cosh}(\xi) \triangleq \text{diag}\{\cosh(\xi_1), \ldots, \cosh(\xi_n)\}, \tag{3}$$

where $\xi = [\xi_1, \ldots, \xi_n]^T \in \mathbb{R}^n$ and $\text{diag}\{\cdot\}$ represents a diagonal matrix. Based on the definitions in (2) and (3), the following inequalities hold $\forall\xi \in \mathbb{R}^n$ (Zhang, Dawson, de Queiroz, & Dixon, 2000):

$$\|\xi\|^2 \geq \sum_{i=1}^{n} \ln(\cosh(\xi_i)) \geq \frac{1}{2}\tanh^2(\|\xi\|),$$
$$\|\xi\| > \|\text{Tanh}(\xi)\|, \qquad \|\text{Tanh}(\xi)\|^2 \geq \tanh^2(\|\xi\|),$$
$$\xi^T\text{Tanh}(\xi) \geq \text{Tanh}^T(\xi)\,\text{Tanh}(\xi),$$
$$\frac{\|\xi\|}{\tanh(\|\xi\|)} \leq \|\xi\| + 1. \tag{4}$$

3. Control development

The control objective is to design an amplitude-limited continuous controller that will ensure that the generalized state $x(t)$ of the input-delayed system in (1) tracks $x_d(t)$ despite uncertainties and additive bounded disturbances in the dynamic model. To quantify the control objective, a tracking error, denoted by $e(x, t) \in \mathbb{R}^n$, is defined as

$$e \triangleq x_d - x. \tag{5}$$

Embedding the control in a bounded trigonometric term (e.g., $\tanh(\cdot)$) is an obvious way to limit the control authority below an a priori limit; however, a difficulty arises in the closed-loop stability analysis with respect to the delay present in the control. Motivated by these stability analysis complexities, and through an iterative analysis procedure, a measurable filtered tracking error is designed which includes additional smooth saturation terms and a finite integral of past control values. Specifically, the filtered tracking error $r\left(e, \dot{e}, e_f, e_z, t\right) \in \mathbb{R}^n$ is defined as

$$r \triangleq \dot{e}(\dot{x}, t) + \alpha \text{Tanh}(e) + \text{Tanh}\left(e_f\right) - e_z(t), \tag{6}$$

where $\alpha \in \mathbb{R}^+$ is a known adjustable gain constant, $e_f(e, r, t) \in \mathbb{R}^n$ is the solution of the auxiliary error filter dynamics given by

$$\dot{e}_f \triangleq \text{Cosh}^2\left(e_f\right)\left(-kr + \text{Tanh}(e) - \gamma \text{Tanh}\left(e_f\right)\right), \tag{7}$$

where $e_f(0) = 0$ and $k, \gamma \in \mathbb{R}^+$ are constant control gains, and $e_z(t) \in \mathbb{R}^n$ denotes the finite integral of past control values, defined as

$$e_z \triangleq \int_{t-\tau}^{t} u(\theta)\, d\theta. \tag{8}$$

From the definition in (8), the finite integral can be upper bounded as $\|e_z\| \leq \zeta_z$, where $\zeta_z \in \mathbb{R}^+$ is a known bounding constant provided the control is bounded.

The open-loop error system can be obtained by taking the time derivative of (6) and utilizing the expressions in (1) and (5) to yield

$$\dot{r} = \ddot{x}_d(t) - f(x, \dot{x}, t) - u(t) - d(t) + \alpha \text{Cosh}^{-2}(e)\, \dot{e}(\dot{x}, t) + \text{Cosh}^{-2}\left(e_f\right) \dot{e}_f\left(e, e_f, r, t\right). \tag{9}$$

From (9) and the subsequent stability analysis, the control input, $u\left(e, e_f, t\right)$, is designed as

$$u \triangleq -k\text{Tanh}\left(e_f\right) + 2\text{Tanh}(e), \tag{10}$$

where k was introduced in (7).[3]

An important feature of the controller given by (10) is its applicability to the case where constraints exist on the available actuator commands. Note that the control law is bounded by the adjustable control gain k, since $\|u\| \leq (k+2)\sqrt{n}$.

The strategy employed to develop the controller in (10) entails several components. One component is the development of the filtered error system in (6) and (7), which is composed of saturated hyperbolic tangent functions designed from the Lyapunov analysis to cancel cross terms. The filtered error system also includes a predictor term (8), which utilizes past values of the control. The motivation for the design of (7) stems from the need to inject a $-kr$ signal into the closed-loop error system, since such terms cannot be directly injected through the saturated controller, and to cancel cross terms in the analysis. The saturated control structure motivates the need for hyperbolic tangent functions in the Lyapunov analysis to yield $-\left\|\text{Tanh}\left(e_f\right)\right\|^2$ terms. The time derivative of the hyperbolic tangent function will yield a $\text{Cosh}^{-2}(e_f)$ term. The design of (7) is motivated by the desire to cancel the $\text{Cosh}^{-2}(e_f)$ term, enabling the remaining terms to provide the desired feedback and cancel nonconstructive terms as dictated by the subsequent stability analysis.

The closed-loop error system is obtained by utilizing (7), (9) and (10) to yield

$$\dot{r} = S(x_d, \dot{x}_d, \ddot{x}_d, t) + \chi\left(e, \dot{e}, e_f, t\right) + k\text{Tanh}\left(e_f\right) - \text{Tanh}(e) - kr\left(e, \dot{e}, e_f, e_z, t\right), \tag{11}$$

where the auxiliary terms $S(x_d, \dot{x}_d, \ddot{x}_d, t) \in \mathbb{R}^n$ and $\chi\left(e, \dot{e}, e_f, t\right) \in \mathbb{R}^n$ are defined as

$$S \triangleq \ddot{x}_d(t) - f(x_d, \dot{x}_d, t) - d(t), \tag{12}$$

$$\chi \triangleq -f(x, \dot{x}, t) + f(x_d, \dot{x}_d, t) + \alpha \text{Cosh}^{-2}(e)\, \dot{e}(\dot{x}, t) - \gamma \text{Tanh}\left(e_f\right). \tag{13}$$

The structure of (11) is motivated by the desire to segregate terms that can be upper bounded by state-dependent terms and terms that can be upper bounded by constants. Using Assumptions 1 and 2, the following inequality can be developed based on the expression in (12):

$$\|S\| \leq \bar{s}, \tag{14}$$

where $\bar{s} \in \mathbb{R}^+$ is a known constant. Using Assumption 3, (4) and (6), the expression in (13) can be upper bounded as (de Queiroz, Hu, Dawson, Burg, & Donepudi, 1997, Appendix A)

$$\|\chi\| \leq \bar{\chi}(\|z\|)\|z\|, \tag{15}$$

where the bounding function $\bar{\chi}(\|z\|) : \mathbb{R} \to \mathbb{R}$ is a positive globally invertible function, and $z\left(e, e_f, r, e_z, P\right) \in \mathbb{R}^{4n+1}$ is defined as

$$z \triangleq \left[e^T \quad \text{Tanh}^T\left(e_f\right) \quad r^T \quad e_z^T \quad \sqrt{P}\right]^T. \tag{16}$$

In (16), $P(t) \in \mathbb{R}^+$ denotes an LK functional defined as

$$P \triangleq \omega \int_{t-\tau}^{t} \left(\int_{s}^{t} \|u(\theta)\|^2\, d\theta\right) ds, \tag{17}$$

where $\omega \in \mathbb{R}^+$ is a known constant.

4. Stability analysis

Theorem 1. *Given the dynamics in (1), the controller in (10) ensures uniformly ultimately bounded tracking provided the adjustable control gains α, γ, k are selected according to the following sufficient conditions:*

$$\alpha > \frac{\psi^2}{4} + 2\omega\tau(2k+1), \quad \gamma > k\omega\tau(k+2), \quad \omega\psi^2 > 2\tau, \tag{18}$$

$$4\beta k_2 \geq \bar{\chi}^2(\bar{\mu})\left(\cosh^{-1}\left(e^{2\bar{\mu}^2}\right) + 1\right)^2, \tag{19}$$

where $\psi \in \mathbb{R}^+$ is an known adjustable positive constant, $\bar{\mu} \in \mathbb{R}$ is defined as $\bar{\mu} \triangleq \max\left\{\bar{d}, \|z(0)\|\right\}$, and $\bar{d} \in \mathbb{R}$ is a subsequently defined positive constant that defines the radius of a ball containing the position tracking errors.

Proof. Let $V_L(z, t) : \mathcal{D} \times [0, \infty) \to \mathbb{R}$ be a continuously differentiable positive-definite functional on a domain $\mathcal{D} \subseteq \mathbb{R}^{4n+1}$, defined as

$$V_L \triangleq \frac{1}{2}r^T r + \sum_{i=1}^{n} \ln(\cosh(e_i)) + \frac{1}{2}\text{Tanh}^T\left(e_f\right)\text{Tanh}\left(e_f\right) + P, \tag{20}$$

which can be bounded using (4) as

$$\phi_1(\|z\|) \leq V_L \leq \phi_2(\|z\|), \tag{21}$$

where the strictly increasing nonnegative functions $\phi_1(\cdot), \phi_2(\cdot) : \mathbb{R}^{4n+1} \to \mathbb{R}$ are defined as $\phi_1(\|z\|) \triangleq \frac{1}{2}\ln(\cosh(\|z\|)), \phi_2(\|z\|) \triangleq \|z\|^2$.

After utilizing (6), (7), and (11), and canceling similar terms, the time derivative of (20) can be expressed as

$$\dot{V}_L = r^T\chi + r^T S - kr^T r - \alpha \text{Tanh}^T(e)\text{Tanh}(e) - \gamma \text{Tanh}^T\left(e_f\right)\text{Tanh}\left(e_f\right) + \text{Tanh}^T(e)\, e_z + \omega\tau\|u\|^2 - \omega\int_{t-\tau}^{t}\|u(\theta)\|^2\, d\theta, \tag{22}$$

[3] To implement the controller in (10), the tracking error $e(\cdot)$ and the integral of past control values $e_z(\cdot)$ should be evaluated first. The signal $e_z(\cdot)$ is considered to be 0 until $t = \tau$. The filtered tracking error $r(\cdot)$ can be evaluated using either the initial condition for $e_f(\cdot)$ ($e_f(0) = 0$ as stated after (7)) or the computed value after the first iteration. The auxiliary signal $e_f(\cdot)$ can be solved online by evaluating $\dot{e}_f(\cdot)$ at each time step using the computed values for $e(\cdot)$ and $r(\cdot)$ and the previous value for $e_f(\cdot)$. Since each of the terms on the right-hand side of (7) is measurable, the solution $e_f(t)$ can be found using any of the numerous numerical integration techniques available in the literature. Once each of the auxiliary error signals has been computed, (10) can be implemented.

where the Leibniz integral rule was applied to determine the time derivative of (17). Using (4), (10), (14), and (15), (22) can be upper bounded by

$$\begin{aligned}\dot{V}_L \leq & -k\|r\|^2 - \alpha\|\mathrm{Tanh}(e)\|^2 - \gamma\left\|\mathrm{Tanh}(e_f)\right\|^2 \\ & + \|r\|\,\bar{\chi}(\|z\|)\|z\| + \|r\|\,\bar{s} + \|\mathrm{Tanh}(e)\|\,\|e_z\| \\ & + k^2\omega\tau\left\|\mathrm{Tanh}(e_f)\right\|^2 + 4\omega\tau\|\mathrm{Tanh}(e)\|^2 \\ & + 4k\omega\tau\left\|\mathrm{Tanh}(e_f)\right\|\,\|\mathrm{Tanh}(e)\| - \omega\int_{t-\tau}^{t}\|u(\theta)\|^2 d\theta. \quad (23)\end{aligned}$$

Young's inequality can be used to upper bound select terms in (23) as

$$\|\mathrm{Tanh}(e)\|\,\|e_z\| \leq \frac{\psi^2}{4}\|\mathrm{Tanh}(e)\|^2 + \frac{1}{\psi^2}\|e_z\|^2, \quad (24)$$

$$\left\|\mathrm{Tanh}(e_f)\right\|\,\|\mathrm{Tanh}(e)\| \leq \frac{1}{2}\left\|\mathrm{Tanh}(e_f)\right\|^2 + \frac{1}{2}\|\mathrm{Tanh}(e)\|^2,$$

where ψ is a known constant. Utilizing the Cauchy–Schwarz inequality, the last integral in (22) can be upper bounded as

$$-\omega\int_{t-\tau}^{t}\|u(\theta)\|^2 d\theta \leq -\frac{\omega}{2\tau}\|e_z\|^2 - \frac{\omega}{2}\int_{t-\tau}^{t}\|u(\theta)\|^2 d\theta. \quad (25)$$

Using (24) and (25), (23) can be upper bounded as

$$\begin{aligned}\dot{V}_L \leq & -k_1\|r\|^2 - \left(\alpha - \frac{\psi^2}{4} - 4\omega\tau\left(\frac{k}{2}+1\right)\right)\|\mathrm{Tanh}(e)\|^2 \\ & - \left(\gamma - 2k\omega\tau - k^2\omega\tau\right)\left\|\mathrm{Tanh}(e_f)\right\|^2 \\ & - \left(\frac{\omega}{2\tau} - \frac{1}{\psi^2}\right)\|e_z\|^2 - k_2\|r\|^2 + \bar{\chi}(\|z\|)\|z\|\,\|r\| \\ & - k_3\|r\|^2 + \bar{s}\|r\| - \frac{\omega}{2}\int_{t-\tau}^{t}\|u(\theta)\|^2 d\theta \quad (26)\end{aligned}$$

where k, introduced in (7) and (10), is split into adjustable constants $k_1, k_2, k_3 \in \mathbb{R}^+$ as $k \triangleq k_1 + k_2 + k_3$. After completing the squares, the expression in (26) can be upper bounded as

$$\begin{aligned}\dot{V}_L \leq & -k_1\|r\|^2 - \left(\alpha - \frac{\psi^2}{4} - 4\omega\tau\left(\frac{k}{2}+1\right)\right)\|\mathrm{Tanh}(e)\|^2 \\ & - \left(\gamma - 2k\omega\tau - k^2\omega\tau\right)\left\|\mathrm{Tanh}(e_f)\right\|^2 \\ & - \left(\frac{\omega}{2\tau} - \frac{1}{\psi^2}\right)\|e_z\|^2 + \frac{\bar{\chi}^2(\|z\|)}{4k_2}\|z\|^2 \\ & - \frac{\omega}{2}\int_{t-\tau}^{t}\|u(\theta)\|^2 d\theta + \frac{\bar{s}^2}{4k_3}. \quad (27)\end{aligned}$$

The inequality

$$\begin{aligned}\int_{t-\tau}^{t}\left(\int_{s}^{t}\|u(\theta)\|^2 d\theta\right)ds & \leq \tau\sup_{s\in[t,\,t-\tau]}\left[\int_{s}^{t}\|u(\theta)\|^2 d\theta\right] \\ & = \tau\int_{t-\tau}^{t}\|u(\theta)\|^2 d\theta\end{aligned}$$

can be used to upper bound (27) as

$$\begin{aligned}\dot{V}_L \leq & -k_1\|r\|^2 - \left(\alpha - \frac{\psi^2}{4} - 4\omega\tau\left(\frac{k}{2}+1\right)\right)\|\mathrm{Tanh}(e)\|^2 \\ & - \left(\gamma - 2k\omega\tau - k^2\omega\tau\right)\left\|\mathrm{Tanh}(e_f)\right\|^2 \\ & - \left(\frac{\omega}{2\tau} - \frac{1}{\psi^2}\right)\|e_z\|^2 + \frac{\bar{\chi}^2(\|z\|)}{4k_2}\|z\|^2 \\ & - \frac{\omega}{2\tau}\int_{t-\tau}^{t}\left(\int_{s}^{t}\|u(\theta)\|^2 d\theta\right)ds + \frac{\bar{s}^2}{4k_3}. \quad (28)\end{aligned}$$

Let $y\left(e, e_f, e_z, r, P\right) \in \mathbb{R}^{4n+1}$ be defined as

$$y \triangleq \left[\mathrm{Tanh}^T(e)\;\; \mathrm{Tanh}^T\left(e_f\right)\;\; e_z^T\;\; r^T\;\; \sqrt{P}\right]^T. \quad (29)$$

By using (16) and (29), (28) can be upper bounded as

$$\dot{V}_L \leq -\beta\|y\|^2 + \frac{\bar{\chi}^2(\|z\|)}{4k_2}\|z\|^2 + \frac{\bar{s}^2}{4k_3}, \quad (30)$$

where the auxiliary constant $\beta \in \mathbb{R}^+$ is defined as

$$\begin{aligned}\beta \triangleq \min\Bigg\{ & k_1,\; \alpha - \frac{\psi^2}{4} - 4\omega\tau\left(\frac{k}{2}+1\right), \\ & \gamma - 2k\omega\tau - k^2\omega\tau,\; \frac{\omega}{2\tau} - \frac{1}{\psi^2},\; \frac{1}{2\tau}\Bigg\}. \quad (31)\end{aligned}$$

If the sufficient conditions in (18) are satisfied, then $\beta > 0$. The conditions in (18) and (19) are solvable for a sufficiently small τ. Provided the following inequality is satisfied,

$$\frac{\bar{\chi}^2(\|z\|)}{4k_2}\|z\|^2 - \beta\|y\|^2 \leq 0, \quad (32)$$

(30) can be expressed as

$$\dot{V}_L \leq -\beta_2\|y\|^2 + \frac{\bar{s}^2}{4k_3}, \quad (33)$$

where $\beta_2 \in \mathbb{R}^+$ is some constant. From the definitions in (16) and (29), and utilizing the fact that $\|y\|^2 \geq \tanh^2(\|z\|)$ from (4), the expression in (32) is satisfied if

$$\left(\frac{\|z\|}{\tanh(\|z\|)}\right)^2 \leq \frac{4\beta k_2}{\bar{\chi}^2(\|z\|)}. \quad (34)$$

Using the properties in (4), a sufficient condition for (34) is

$$(\|z\|+1)^2 \leq \frac{4\beta k_2}{\bar{\chi}^2(\|z\|)}. \quad (35)$$

We can now utilize the lower bound on $V_L(z,t)$ from (21), to state that

$$\|z\| \leq \cosh^{-1}(\exp(2V_L)); \quad (36)$$

hence, a sufficient condition for (35) can be obtained as

$$\bar{\chi}^2\left(\cosh^{-1}(\exp(2V_L)) + 1\right)^2 \leq 4\beta k_2. \quad (37)$$

If condition (37) is satisfied, then, from (4), the expression in (33) can be rewritten as

$$\dot{V}_L \leq -\phi_3(\|z\|) + \frac{\bar{s}^2}{4k_3}, \quad (38)$$

where the strictly increasing nonnegative function $\phi_3 : \mathbb{R}^{4n+1} \to \mathbb{R}$ is defined as $\phi_3(\|z\|) \triangleq \beta_2\tanh^2(\|z\|)$. Given (21), and (38), $z(\cdot)$ (as well as $e(\cdot)$ and $r(\cdot)$ via the definition in (16) and standard linear analysis) is uniformly ultimately bounded (Corless & Leitmann, 1981) in the sense that

$$\|e(t)\| \leq \|z(t)\| < \bar{d}, \quad \forall t \geq T\left(\bar{d},\; \|z(0)\|\right), \quad (39)$$

provided the sufficient conditions in (18) and the inequality in (37) are satisfied. In (39), $\bar{d}$ defines the radius of a ball containing the position tracking errors, selected according to Corless and Leitmann (1981)

$$\bar{d} > \left(\phi_1^{-1} \circ \phi_2\right) \left(\phi_3^{-1} \left(\frac{\bar{s}^2}{4k_3}\right)\right), \tag{40}$$

and $T\left(\bar{d},\ \|z(0)\|\right) \in \mathbb{R}$ is a positive constant that denotes the ultimate time to reach the ball (Corless & Leitmann, 1981):

$$T \triangleq \begin{cases} 0 & \|z(0)\| \leq \left(\phi_2^{-1} \circ \phi_1\right)\left(\bar{d}\right) \\ \dfrac{\phi_2\left(\|z(0)\|\right) - \phi_1\left(\left(\phi_2^{-1} \circ \phi_1\right)\left(\bar{d}\right)\right)}{\phi_3\left(\phi_2^{-1} \circ \phi_1\right)\left(\bar{d}\right) - \frac{\bar{s}^2}{4k_3}} & \|z(0)\| > \left(\phi_2^{-1} \circ \phi_1\right)\left(\bar{d}\right). \end{cases}$$

From (21) and (39), a final sufficient condition for (37), given in (19), can be expressed in terms of either the initial conditions of the system or the ultimate bound. □

Remark 2. Based on (38), the size of the ultimate bound in (40) can be made smaller by selecting k_3 larger. For arbitrarily large delays or arbitrarily large initial conditions, the control gains required to satisfy the sufficient gain conditions in (19) may demand torque that is not physically deliverable by the system (i.e., the gain k may be required to be larger than the saturation limit of the actuator). The gain condition in (19) is directly influenced by the bound given in (15), which results from the bounds derived in Assumption 3. For example, if f is globally Lipschitz, then the upper bound on χ reduces to a constant times the state, and a local condition on the state z can be determined as $\|z(0)\| \leq \sqrt{4\beta k_2/\bar{\chi}} - 1$, which can be enlarged by increasing k_2 (up to a point based on the actuator constraints). Given the current more general bound for χ from Assumption 3, a simplified closed-form initial condition bound cannot be derived. However, given an upper bound on the disturbance, an upper bound on the time delay, and the initial conditions, (19) and (31) can be used to determine the sufficient gain βk_2, if possible, based on the actuator limit. This result does not satisfy the standard semi-global result because, under the consideration of input constraints, k cannot be arbitrarily increased and consequently cannot satisfy all initial conditions. This outcome is not surprising from a physical perspective in the sense that such demands may yield cases where the actuation is insufficient to stabilize the system.

Remark 3. From the sufficient gain conditions in (18) and (19), the admissible values for the known input delay can be determined as $\tau = \min\left\{\frac{\omega\psi^2}{2} - \frac{\zeta(\bar{\mu})}{4k_2}, \frac{\gamma}{k\omega(k+2)} - \frac{\zeta(\bar{\mu})}{4k_2}, \frac{\alpha - \frac{\psi^2}{4}}{2\omega(2k+1)} - \frac{\zeta(\bar{\mu})}{4k_2}, \frac{\zeta(\bar{\mu})}{2k_2}\right\}$, where $\zeta(\bar{\mu})$ denotes the right-hand side of (19). Independent of the actuator saturation limits, α and γ can be made arbitrarily large, and the remaining conditions can be selected according to (18) and (19).

5. Simulation

The controller in (10) was simulated for a two-link planar manipulator. The manipulator can be modeled as an Euler–Lagrange system with the following dynamics:

$$M(q)\ddot{q} + V_m(q, \dot{q})\dot{q} + F(\dot{q}) + \tau_d(t) = \tau_c(t - \tau), \tag{41}$$

where $M(q) \in \mathbb{R}^{2\times 2}$ denotes the inertia matrix, $V_m(q, \dot{q}) \in \mathbb{R}^{2\times 2}$ denotes the centripetal-Coriolis matrix, $F(\dot{q}) \in \mathbb{R}^2$ denotes friction, and $\tau_d(t) \in \mathbb{R}^n$ denotes an external disturbance. Additionally, $q(t), \dot{q}(t), \ddot{q}(t) \in \mathbb{R}^2$ denote the link position, velocity,

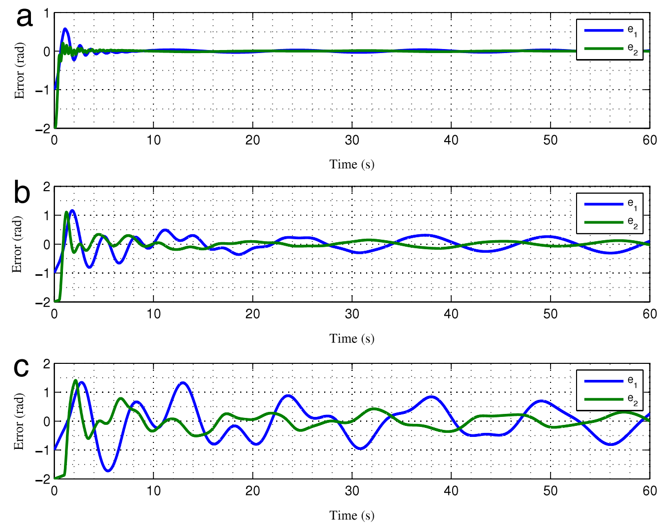

Fig. 1. Tracking error versus time for (a) 100 ms input delay, (b) 500 ms input delay, (c) 1 s input delay.

and acceleration, and $\tau_c(t - \tau) \in \mathbb{R}^2$ denotes the control torque. In (41), $M(q) \triangleq \begin{bmatrix} p_1 + 2p_3c_2 & p_2 + p_3c_2 \\ p_2 + p_3c_2 & p_2 \end{bmatrix}$, $V_m(q, \dot{q}) \triangleq \begin{bmatrix} -p_3s_2\dot{q}_2 & -p_3s_2(\dot{q}_1 + \dot{q}_2) \\ p_3s_2\dot{q}_1 & 0 \end{bmatrix}$, and $F(\dot{q}) \triangleq \begin{bmatrix} f_{d1} & 0 \\ 0 & f_{d2} \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$, where $p_1 = 3.473\ \text{kg}\cdot\text{m}^2$, $p_2 = 0.196\ \text{kg}\cdot\text{m}^2$, $p_3 = 0.242\ \text{kg}\cdot\text{m}^2$, $f_{d1} = 5.3\ \text{N m s}$, $f_{d2} = 1, 1\ \text{N m s}$, c_2 denotes $\cos(q_2)$, and s_2 denotes $\sin(q_2)$. The dynamics in (41) can be rewritten in the form of (1), where $x(t) = \begin{bmatrix} q_1(t), & q_2(t) \end{bmatrix}^T$, $f(x, \dot{x}, t) \triangleq -M^{-1}(q)\left(V_m(q, \dot{q})\dot{q} + F(\dot{q})\right)$, $u(t - \tau) \triangleq M^{-1}(q)\tau_c(t - \tau)$, and $d(t) \triangleq -M^{-1}(q)\tau_d(t)$.[4] An explicit bound for $M(q)$ can be found, and thus a bound for $\tau_c(t - \tau)$ is obtainable a priori. The disturbance terms were selected as $\tau_{d1} = 0.5\sin\left(\frac{t}{5}\right)$, and $\tau_{d2} = 0.1\sin\left(\frac{t}{5}\right)$. The desired trajectories for links 1 and 2 for all simulations were selected as

$$q_{d1}(t) = 1.5\sin(t/2)\ \text{rad}$$
$$q_{d2}(t) = 0.5\sin(t/4)\ \text{rad}.$$

The initial conditions for the manipulator were selected as stationary with a significant offset from the initial conditions of the desired trajectory as $\begin{bmatrix} q_1 & q_2 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$ rad. For comparison, the simulation was completed using various values of input delay, ranging from 100 ms to 1 s. For each case, the actuation torque was limited to $\tau_1 \leq 20$ N, $\tau_2 \leq 10$ N.

Fig. 1 illustrates the tracking errors associated with each of the input delay cases. As the delay magnitude is increased, the performance degrades and the tracking error bound increases. Fig. 2 shows that, even with a large input delay in the system, the proposed controller is able to ensure that the control torque does not exceed the actuator limits (as specified by the controller gains) while ensuring the boundedness of the tracking error.

Remark 4. The controller presented in (10) can also be extended to standard nonlinear Euler–Lagrange systems where the design of the error systems and controller follow similarly to the presented method and two additional LK functionals are included to cancel cross terms associated with uncertainties in inertia. This extension

[4] To transform (41) into the general form of (1), we assume that the Inertia matrix is known in this case. For a complete analysis of a Euler–Lagrange system with unknown Inertia matrix, we refer the reader to the conference version of this work in Fischer et al. (2011).

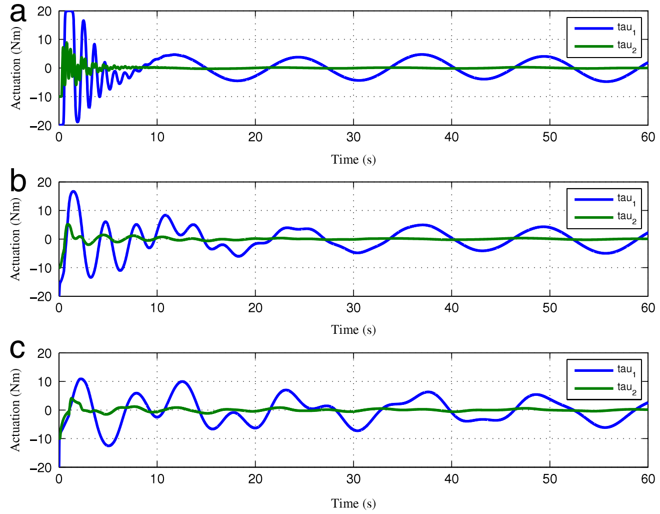

Fig. 2. Control torque $\tau(t)$ versus time for (a) 100 ms input delay, (b) 500 ms input delay, (c) 1 s input delay.

can be considered as an application of the presented method. Specifically, (6) can be modified to accommodate the uncertain inertia effects in the dynamics as

$$r \triangleq \dot{e} + \alpha \text{Tanh}\,(e) + \text{Tanh}\,(e_f) - Be_z,$$

where $B \in \mathbb{R}^{n \times n}$ is a symmetric positive-definite constant gain matrix. The remaining error system development follows identically from the proposed result, i.e., (7), (8) and (10) are designed the same. For additional details, see Fischer et al. (2011).

6. Conclusion

A continuous saturated controller is developed for uncertain nonlinear systems which include input delays and sufficiently smooth additive bounded disturbances. The bound on the control is known a priori and can be adjusted by changing the feedback gains. The saturated controller is shown to guarantee uniformly ultimately bounded tracking provided the delay is sufficiently small, and a numerical example demonstrating the performance of the control design is presented. Extending the result for uncertain time-varying time delays will enhance the applicability of the controller; this is the focus of on-going efforts.

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N. Fischer received his Ph.D. in 2012 from the Department of Mechanical and Aerospace Engineering at the University of Florida (UF). His graduate studies were conducted with the Nonlinear Controls and Robotics Group under the advisement of Dr. Warren Dixon. He is a recipient of the 2012 Graduate Attribute in Professional Excellence from the UF College of Engineering and the 2012 Outstanding Graduate Researcher Award from the UF Department of Mechanical and Aerospace Engineering. He currently works as a Technical Lead Systems Engineer at Bluefin Robotics. His research interests include the development of Lyapunov-based control methods for uncertain nonlinear systems with specific application interests in autonomous underwater vehicles, mobile robotics, and financial markets.

A. Dani received his Ph.D. degree in 2011 from the Department of Mechanical and Aerospace Engineering at the University of Florida. He is currently working as a postdoctoral research associate in the Aerospace Engineering department at the University of Illinois at Urbana-Champaign. His research interests are in the area of nonlinear estimation theory, control of uncertain nonlinear systems with time delay, vision-based control, and multi-agent systems. He holds two patents in the area of vision-based estimation. His doctoral dissertation has won best dissertation award from the Mechanical and Aerospace Engineering department at the University of Florida.

N. Sharma received his Ph.D. in 2010 from the Department of Mechanical and Aerospace Engineering at the University of Florida. He is a recipient of 2009 O. Hugo Schuck Award and Best Student Paper Award in Robotics at the 2009 ASME Dynamic Systems and Controls Conference. He was also a finalist for the Best Student Paper Award at the 2008 IEEE Multi-Conference on Systems and Control. After his doctoral studies, he was an Alberta Innovates (Health Solutions) postdoctoral fellow in the Department of Physiology at the University of Alberta, Edmonton, Canada. In 2012, he joined the Department of Mechanical Engineering and Material Science at the University of Pittsburgh as an assistant professor. His research interests include intelligent and robust control of functional electrical stimulation (FES), modeling, optimization, and control of FES-elicited walking, and control of uncertain nonlinear systems with input and state delays.

W.E. Dixon received his Ph.D. degree in 2000 from the Department of Electrical and Computer Engineering from Clemson University. After completing his doctoral studies he was selected as an Eugene P. Wigner Fellow at Oak Ridge National Laboratory (ORNL). In 2004, he joined the faculty of the University of Florida in the Mechanical and Aerospace Engineering Department, where he currently is the Charles Taylor Faculty Fellow (2012–2015) and holds a University of Florida Research Foundation Professorship (2012–2015). He has published 3 books, an edited collection, 9 chapters, and over 250 refereed journal and conference papers related to Lyapunov-based control techniques for uncertain nonlinear systems. His work has been recognized by the 2011 American Society of Mechanical Engineers (ASME) Dynamics Systems and Control Division Outstanding Young Investigator Award, 2009 American Automatic Control Council (AACC) O. Hugo Schuck Award, 2006 IEEE Robotics and Automation Society (RAS) Early Academic Career Award, an NSF CAREER Award (2006–2011), 2004 DOE Outstanding Mentor Award, and the 2001 ORNL Early Career Award for Engineering Achievement. He is a senior member of IEEE. He serves on several IEEE CSS and ASME technical committees, and is a member of numerous conference program and organizing committees. He served as an appointed member to the IEEE CSS Board of Governors (BoG) in 2008, and he now serves as the Director of Operations for the Executive Committee of the BoG.